Derivatives in Economics

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Derivatives appear in economics! The term "**marginal**" are used to denote the rate of change of a quantity with respect to the variable on which it depends. Therefore, the "marginal" of a function means the "derivative" of a function! For example,

- **Marginal Cost**: Suppose that $C(x)$ is the total cost function for a manufacturer to produce x units of a commodity. Then the *marginal cost of production* is $C'(x)$ which can be used to estimate the extra cost of producing one more unit

$$\Delta C = C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1}.$$

- **Marginal Revenue, Marginal Profit**: Suppose that $R(x)$ is the revenue generated when x units of a commodity is produced and sold, and $P(x)$ is the corresponding profit. Then, *marginal revenue* $R'(x)$ approximates the additional revenue of producing one more unit, $\Delta R = R(x+1) - R(x)$, and *marginal profit* $P'(x)$ approximates $\Delta P = P(x+1) - P(x)$.
- **Marginal Utility**: Let $U(x)$ be the utility of consuming x units of a commodity. Then, we use *marginal utility* $U'(x)$ to estimate the additional utility of consuming one more unit.

Exercise: The weekly total cost for manufacturing x sets of Pulsar 40-in television is

$$C(x) = 10^{-4}x^3 - 0.06x^2 + 500x + 10000$$

1. Use marginal cost to estimate the extra cost of producing the 11^{th} Pulsar TV, $C(11) - C(10)$.

We use $C'(10)$ to estimate $C(11) - C(10)$.

$C'(x) = 3\times10^{-4}x^2 - 0.12x + 500$, $C'(10) = 498.83$

2. When is marginal cost decreasing? When is it increasing? Try to explain this economic phenomena. $C'(x) = 3\times10^{-4}x^2 - 0.12x + 500$. $C''(x) = 6\times10^{-4}x - 0.12$.

$C''(x) < 0$ for $0 < x < 200$ $\Rightarrow$ $C'(x)$ is decreasing for $0 < x < 200$.

(This is called the "economy of scale".)

$C''(x) > 0$ for $x > 200$ $\Rightarrow$ $C'(x)$ is increasing for $x > 200$.

3. If the weekly demand for Pulsar TV is described as $p = -0.125x + 600$, $0 \le x \le 4800$ where p is the unit price and x is the quantity demanded. How many TV sets should be produced weekly to maximize profit?

The profit is $\pi(x) = x \cdot p(x) - C(x) = -0.125x^2 + 600x - 10^{-4}x^3 + 0.06x^2 - 500x - 10000$

To find the maximum value of $\pi(x)$ on $[0, 4800]$, we find critical numbers of $\pi(x)$ in $(0, 4800)$.

Solve $\pi'(x) = -3\times10^{-4}x^2 - 0.13x + 100 = 0$ $\Rightarrow$ $x = 400$ or $-\frac{2500}{3}$.

$\pi(400) = 13200$, $\pi(0) = -10000$, $\pi(4800) < 0$.

Hence the maximum profit is 13200 which is obtained by producing 400 TV sets weekly.

Economists usually use different scales of measurement when counting production quantity (in thousands vs. in millions), or pricing (using different currency). This creates a problem for using the derivative $f'(p)$ to describe the sensitivity of the demand y for a certain product to the price p charged for the product: $y = f(p)$.
Instead, economists use the **point elasticity of demand** to describe price sensitivity, which is defined as the percentage change of quantity demanded divided by the percentage change in prices. When the units are divisible, the **point elasticity of demand** is

$$\epsilon = \lim_{\Delta p \to 0} \frac{\frac{y(p+\Delta p)-y(p)}{y(p)}}{\frac{\Delta p}{p}} = \frac{p}{y} \cdot \frac{dy}{dp}$$

A commodity is said to be **elastic** if $\epsilon < -1$. A commodity is **inelastic** if $-1 < \epsilon < 0$.

Exercise:

1. Consider a demand function $y(p) = a + bp$ for $0 \le p \le -\frac{a}{b}$ where $b < 0$. Find the values of p such that the point elasticity is between 0 and -1. Find the values of p such that the point elasticity is smaller than -1.

$\varepsilon = \frac{dy}{dp} \cdot \frac{p}{y} = \frac{b \cdot p}{y} = \frac{b \cdot p}{a+bp}$.

$\boxed{\text{Case 1}}$

$-1 < \varepsilon < 0 \iff -1 < \frac{b \cdot p}{a+bp} < 0$

$\iff -a-bp < bp < 0$

$\because b<0$, $-a-bp < bp \iff -a < 2bp$

$\Rightarrow p < \frac{-a}{2b}$.

$\because b<0$, $bp<0 \Rightarrow p>0$

$\Rightarrow 0 < p < \frac{-a}{2b}$.

$\boxed{\text{Case 2}}$

$\varepsilon < -1 \iff \frac{bp}{a+bp} < -1 \iff bp < -a-bp$

$\because b<0$, $bp < -a-bp \iff 2bp < -a$

$\Rightarrow p > \frac{-a}{2b} \Rightarrow \frac{-a}{2b} < p < \frac{-a}{b}$.

2. What is the point elasticity when demand takes the form $y = Kp^{-r}$ where K and r are positive constants? Does it depend on price p?

$\varepsilon = \frac{dy}{dp} \cdot \frac{p}{y} = \left(k \cdot (-r) \cdot p^{-r-1}\right) \times \frac{p}{kp^{-r}} = -r$.

The point elasticity is a contant function $\varepsilon(p) = -r$ which doesn't depend on price p.

(Conversely, you can prove that if $\varepsilon(p) = -r$, a constant function, then $y(p) = k \cdot p^{-r}$ for some constant $k > 0$.)

3. Show that if a commodity is inelastic, then an increase in price leads to an increase in total revenue $y(p) \cdot p$. On the other hand, an increase in price leads to an decrease in total revenue $y(p) \cdot p$ when the commodity is elastic.

$\frac{d}{dp}(y(p) \cdot p) = y'(p) \cdot p + y(p) = y(p)\left[\frac{y'}{y} \cdot p + 1\right] = y \cdot (\varepsilon(p) + 1)$.

If $-1 < \varepsilon(p) < 0$, then $\frac{d}{dp}(y(p) \cdot p) > 0$ and the revenue $y(p) \cdot p$ is increasing in terms of p.

If $\varepsilon(p) < -1$, then $\frac{d}{dp}(y(p) \cdot p) < 0$ and the revenue $y(p) \cdot p$ is decreasing in terms of p.